Errors of spectral estimators in ARMA processes

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RESUME

Notre étude porte sur le problème suivant : Evaluer l'erreur qui apparaît sur un modèle ARMA lorsque la densité spectrale du processus est approximée par une méthode estimative.

Nous estimons l'erreur introduite quand nous évaluons une erreur de covariance pour des estimations spectrales ; de plus nous la comparons avec deux autres méthodes apparues dans la littérature.

SUMMARY

Our purpose is the study of the following problem : to evaluate the error aWppearing when a spectrum density (SPD) of an autoregressive moving average process (ARMA) is approximated by an estimative method.

Finally, we estimate the error introduced when evaluating a covariance error (CVE) for spectral estimators (SPE) ; in addition we compare it with two other methods appearing in the literature.

By using the periodogram for ARMA processes, we simplify the calculation of the SPE. It's clear that we don't need to care about the errors of the AR when evaluating the errors of spectral estimations.

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1 - INTRODUCTION

In AR processes, spectral analysis has been realized by theoretical establishment and development of computer algorithms [1-3].

Random time series have been fitted to ARMA processes, and auto and cross spectra have been estimated by several methods. Recently, complex evaluations and incomplete theories have been gradually solved in ARMA processes. Akaike proposed a canonical correlation method by fitting random time series to multi-ARMA processes [4]. The calculation procedures based on the mximum likelihood method need an excessive computation time. Several authors points out that the SPE for ARMA processes might be computed by using only AR parameters, i.e. without using MA. Tokumaru and Takeyasu proposed a method for estimating spectra which only uses AR parameters in ARMA processes [5].

We proposed the deduced expression of SPD matrix by means of a state model for estimators of auto and cross spectra [7]. This equation does not contain moving average parameters (MA). Thus, we don't need to solve nonlinear simultaneous equations for finding MA parameters.

In this paper, we will analize errors of the SPE matrix $S(\omega)$ in a state space model. Here, we deal only with ARMA processes in one variable, since the multivariable case is more involved but can be solved with the same methods. A SPD matrix was deduced as for $S(\omega) = Q(\omega) + Q'(\omega)$ (where ' means transpose). It is well known that SPE are unbiased estimators, but the evaluation of its errors is still an open problem. We will try to derive mathematical equations to look for errors of SPE. And also we will compare the errors of our method with errors of the method based on the Yule-Walker equation and with the errors of the method based on high order AR approximation for ARMA processes.

2 - THE STATISTICAL ANALYSIS OF ERRORS

Previously, we have deduced SPE in ARMA processes. In this section we discuss about errors of these SPE. The periodogram is introduced in order to simplify evaluations.

2 - 1. The estimations of SPD in ARMA models

In this section, we will recall same facts about SPD of ARMA processes. Let y_s be time series, we write them :

(2.1) $$y_s = y_1 + y_2 + \dots + y_n$$

The SPD in time series y_s have been deduced as follows [5], [7],

(2.2) $$S(\omega) = Q(\omega) + Q(-\omega) - R_0$$

with

(2.3) $$R_\ell \equiv E[y_s \cdot y_{s+\ell}]$$

and

(2.4) $$Q(\omega) = \frac{\sum_{k=1}^{n} \sum_{\ell=0}^{n-k} \alpha_\ell \, R_{n-\ell-k} \, e^{j\omega k}}{\sum_{\ell=0}^{n} \alpha_\ell \, e^{j\omega(n-\ell)}}$$

where α_ℓ is an AR coefficient and $\alpha_0 = 1$ and $j = \sqrt{-1}$ (See Appendix A1).

We can write SPE as follows :

(2.5) $$\hat{S}(\omega) = \hat{Q}(\omega) + \hat{Q}(-\omega) - \hat{R}_0$$

where

(2.6) $$\hat{Q}(\omega) = \frac{\sum_{k=1}^{M} \sum_{\ell=0}^{M-k} \hat{\alpha}_\ell \, \hat{R}_{M-\ell-k} \, e^{j\omega k}}{\sum_{\ell=0}^{M} \hat{\alpha}_\ell \, e^{j\omega(M-\ell)}}$$

with

$$\hat{R}_\ell = \frac{1}{N} \sum_{i=1}^{N-\ell} y_i \, y_{i+\ell}$$

and satisfies

(2.7)
$$\hat{R}_{\ell} = \hat{R}_{-\ell}$$

where N is the sample size.

$\hat{\alpha}_{\ell}$, $\hat{Q}(w)$ and $\hat{R}_{\ell}$ are estimators. We try to deduced mathematical error of SPE in this work.

2 - 2. Introducing the periodogram

We study the approximate properties of SPE. Even if we change the definition of SPD and auto-covariance like this section, influences may be neglected.

We deduce several important equations in this section as follows. SPD $f(\omega)$ is :

(step 1) $$f(\omega) = \frac{1}{2\pi} \{Q(\omega) + Q(-\omega) - R_0\}$$

The error ($\Delta\boldsymbol{\alpha}$) of estimators $\hat{\boldsymbol{\alpha}}$ is :

(step 2) $$\mathbf{F}\Delta\boldsymbol{\alpha} \simeq -\Delta\mathbf{f} - \Delta\mathbf{F}\boldsymbol{\alpha} \simeq -(\mathbf{R}'\hat{\mathbf{r}} + \mathbf{R}'\hat{\mathbf{R}})\boldsymbol{\alpha}$$

and the error (ΔQ) of $\hat{Q}(\omega)$ is :

(step 3) $$\Delta Q(\omega) \simeq \mathbf{H}'(\omega)\ \Delta\boldsymbol{\alpha} + \int_{-\pi}^{\pi} T(s,\omega)\ I_N(s)ds - Q(\omega)$$

where $I_N(s)$ is the periodogram.

The error ($\Delta f(\omega)$) of SPE $\hat{f}(\omega)$ is written

(step 4) $$\Delta f(\omega) \simeq \int_{-\pi}^{\pi} G(s,\omega)\ I_N(s)ds - f(\omega).$$

2.2.1. Step 1 : the definition of $f(\omega)$

We define the following equation for SDP $f(\omega)$.

$$f(\omega) = \frac{1}{2\pi} \sum_{i=-\infty}^{\infty} R_i \, e^{-ji\omega}$$

(2.8) where R_i is a scalar and

$$f(\omega) = \frac{1}{2\pi} \{Q(\omega) + Q(-\omega) - R_0\}$$

where $Q(\omega)$ and $B(\omega)$ are

$$Q(\omega) = \frac{1}{B(\omega)} \sum_{k=0}^{n-1} \sum_{i=0}^{n-k-1} \alpha_k R_i \, e^{j\omega(n-k-i)} \tag{2.9}$$

and

$$B(\omega) = \sum_{i=0}^{n} \alpha_i \, e^{j\omega(n-i)}. \tag{2.10}$$

2.2.2. Step 2 : the error of estimators $\hat{\alpha}$

Assume that true orders n of ARMA processes are known. We define following equations by using auto-covariances $\{R_\ell\}$ and AR coefficients $\{\alpha_i\}$

$$\boldsymbol{\alpha} = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$$

$$\mathbf{r}_\ell = (R_\ell, R_{\ell+1}, \ldots, R_L)$$

$$\mathbf{R} = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}) \tag{2.11}$$

$$\mathbf{F} = \mathbf{R}'\mathbf{R}$$

$$\mathbf{f} = \mathbf{R}'\mathbf{r}_n$$

where n is the true order.

When we substitute $\hat{R}_\ell$ for R_ℓ, we obtain estimators like $\hat{\boldsymbol{\alpha}}$, $\hat{\mathbf{r}}_\ell$ and $\hat{\mathbf{F}}$. These errors of estimators are $\Delta\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}$, $\Delta\mathbf{R} = \hat{\mathbf{R}}-\mathbf{R}$ and $\Delta\mathbf{F} = \hat{\mathbf{F}}-\mathbf{F}$. If N is large enough, then these errors are sufficiently small by the asymptotic theory of statistics. Therefore terms over second order can be neglected. We obtain the following equation for the error of $\hat{\boldsymbol{\alpha}}$

$$(2.12) \qquad \mathbf{F}\Delta\boldsymbol{\alpha} \simeq -\Delta\mathbf{f} - (\Delta\mathbf{F})\boldsymbol{\alpha}$$

$$\simeq -(\mathbf{R}'\hat{\mathbf{r}} + \mathbf{R}'\hat{\mathbf{R}})\boldsymbol{\alpha}$$

where ' means transpose.

2.2.3. Step 3 : the error of $\hat{Q}(\omega)$

Let's introduce the periodogram $I_N(s)$. $I_N(s)$ is defined as follows :

$$(2.13) \qquad I_N(s) = \frac{1}{2\pi N} \left| \sum_{i=1}^{N} y_i \, e^{-jis} \right|^2 \qquad (|s| \le \pi)$$

In other word, $\hat{R}_k$ is defined as

$$(2.14) \qquad \hat{R}_k = \int_{-\pi}^{\pi} I_N(s)\, e^{jks}\, ds.$$

The ith element of eq. (2.12) is given by

$$(2.15) \qquad (\mathbf{F}\Delta\boldsymbol{\alpha})_i \simeq -\int_{-\pi}^{\pi} I_N(s) \sum_{\ell=1}^{L+1} R_{\ell-2+i} \sum_{t=0}^{n} \alpha_t \, e^{j(\ell+n-1-t)s}\, ds.$$

After expansion $\Delta Q(\omega) = \hat{Q}(\omega) - Q(\omega)$ by Taylor expansion, we will chose only the first term. Then we get

$$(2.16) \qquad \Delta Q(\omega) \simeq \frac{1}{B(\omega)} \sum_{k=0}^{n-1} \{ \sum_{i=0}^{n-k-1} (\Delta\alpha_k R_i + \alpha_k \hat{R}_i - \alpha_k R_i)\, e^{j\omega(n-k-i)} - \frac{Q(\omega)}{B(\omega)} \sum_{i=0}^{n} \Delta\alpha_i \, e^{j\omega(n-i)} \qquad (\Delta\alpha_0 = 0)$$

$\overline{\Delta Q}(\omega)$ becomes, by eq. (2.14) :

$$\text{(2.17)}\qquad \overline{\Delta Q}(\omega) = \mathbf{H}'(\omega)\overline{\Delta\boldsymbol{\alpha}} + \int_{-\pi}^{\pi} T(s,\omega)\, I_N(s)\, ds - Q(\omega)$$

where $\mathbf{H}'(\omega) = (H_n(\omega), H_{n-1}(\omega), \ldots, H_1(\omega))$.

And then H_k and $T(s,\omega)$ are

$$\text{(2.18)}\qquad H_k = \frac{1}{B(\omega)} \{ \sum_{i=0}^{n-k-1} R_i\, e^{j\omega(n-k-i)}(1 - \delta_{n,k}) - Q(\omega)e^{j\omega(n-k)}$$

$$\text{(2.19)}\qquad \begin{aligned} T(s,\omega) &= \frac{1}{B(\omega)} \sum_{k=0}^{n-1} e^{jks} \sum_{i=0}^{n-k-1} \alpha_i\, e^{j\omega(n-k-i)} \\ &\overset{\Delta}{=} \sum_{k=0}^{n-1} e^{jks}\, J_k(\omega) \end{aligned}$$

where $J_k(\omega) = \frac{1}{B(\omega)} \sum_{i=0}^{n-k-1} \alpha_i\, e^{j\omega(n-k-1)}$.

2.2.4. Step 4 : the error of $\hat{f}(\omega)$

We try to represent for the first term of eq. (2.17) by the periodogram. $\mathbf{H}'\mathbf{F}^{-1}(\omega)$ is given by $\mathbf{H}'\mathbf{F}^{-1} = \mathbf{M}'(\omega) = (M_1(\omega), M_2(\omega), \ldots, M_n(\omega))$. A new vector $\mathbf{M}'(\omega)$ is defined. We can deduce the following equation from eq. (2.15),

$$\text{(2.20)}\qquad \mathbf{H}'(\omega)\overline{\Delta\boldsymbol{\alpha}} = \mathbf{M}'(\omega)\mathbf{F}\overline{\Delta\boldsymbol{\alpha}} \simeq -\int_{-\pi}^{\pi} I_N(s)\, K(s,\omega)ds$$

where

$$\text{(2.21)}\qquad K(s,\omega) = \sum_{k=1}^{n} \grave{M}_k(\omega) \sum_{\ell=1}^{L+1} R_{\ell-2+k} \sum_{t=0}^{n} \alpha_t\, e^{j(\ell+n-1-t)s}.$$

The error $\overline{\Delta f}(\omega) = \hat{f}(\omega) - f(\omega)$ of SPE which we are looking for is

$$\Delta f(\omega) = \frac{1}{2\pi} \{\Delta Q(\omega) + \Delta Q(-\omega) - \hat{R}_0 + R_0\}$$

(2.22)

$$\simeq \int_{-\pi}^{\pi} G(s,\omega)\, I_N(s)ds - f(\omega)$$

and we can represent as follows

(2.23) $$2\pi\, G(s,\omega) = - K(s,\omega) + T(s,\omega) + [-K(s,\omega) + T(s,\omega)]-1.$$

2 - 3. The covariance of error (CVE)

We intend to deduce the CVE of SPE. We can say as

$$E[I_N(s)] \simeq f(\omega)$$

and also

$$\int_{-\pi}^{\pi} G(s,\omega)\, f(s)ds = f(\omega).$$

The CVE of SPE can be deduced at any angular frequencies of ω and η as follows :

$$E[\Delta f(\omega)\, \Delta f(\eta)] \simeq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(s,\omega)\, G(t,\eta) \operatorname{cov}[I_N(s), I_N(t)]\, dsdt$$

(2.24)

$$+ \{ \int_{-\pi}^{\pi} G(s,\omega)\, E[I_N(s)]ds - f(\omega)\}\{ \int_{-\pi}^{\pi} G(t,\eta)\, E[I_N(t)]dt - f(\eta)\}$$

where the second terms of this equation can be neglected because they are of order N^{-2}

$$\simeq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(s,\omega)\, G(t,\eta) \operatorname{cov}[I_N(s), I_N(t)]\, dsdt$$

(See Appendix A2).

According to the general theory as to the periodogram, covariance is given by

$$\operatorname{cov}[I_N(s),\ I_N(t)] = 2\pi N^{-1}\ f(s)\ f(t)\ [\delta(s+t) + \delta(t-s)]$$
(2.25)
$$+\ N^{-1}\ f(s)\ f(t)\ K_4\ \sigma^{-4}.$$

In this equation, $\delta(\cdot)$ is Dirac's delta function and K_4 is the fourth cumulants of innovation of ARMA processes and σ^2 means variance. We can deduce the following equation by eqs. (2.24) and (2.25) :

$$NE[\frac{\Delta f(\omega)}{f(\omega)} \cdot \frac{\Delta f(\eta)}{f(\eta)}] = k_4\ \sigma^{-4} + \frac{2\pi}{f(\omega)\ f(\eta)} \int_{-\pi}^{\pi} [G(s,\omega)\ G(s,\eta)$$
(2.26)
$$+\ G(s,\omega)\ G(-s,\eta)]\ f^2(s)ds.$$

Using eqs. (2.9), (2.19), (2.21) and (2.23), we obtain the following equation :

(2.27) $$G(s,\omega) = \sum_{k=-L-n}^{L+n} g_k(\omega)\ e^{jk\omega} \qquad (g_{-k}(\omega) = g_k(-\omega))$$

where $g_k(\omega)$ is

$$g_0(\omega) = \frac{1}{2\pi}\{J_0(\omega) + J_0(-\omega) - \sum_{k=1}^{n} M_k(\omega)\ R_{k-1}\ \alpha_n - \sum_{k=1}^{n} M_k(-\omega)\ R_{k-1}\alpha_n{-1}\},$$

(2.28)
$$g_k(\omega) = \begin{cases} \dfrac{1}{2\pi}\ \{J_k(\omega) - \displaystyle\sum_{t=0}^{k} \alpha_{n-t} \sum_{i=1}^{k} R_{i-1+k-t}\ M_i(\omega)\}\ ; & 1 \leq k \leq n-1 \\ \dfrac{-1}{2\pi} \displaystyle\sum_{t=0}^{n} \alpha_t \sum_{i=1}^{n} R_{i+t-1+k-n}\ M_i(\omega)\ ; & n \leq k \leq L \\ \dfrac{-1}{2\pi} \displaystyle\sum_{t=0}^{L+n-k} \alpha_t \sum_{i=1}^{n} R_{i+t-1+k-n}\ M_i(\omega)\ ; & L+1 \leq k \leq L+n. \end{cases}$$

When we substitute eq. (2.26) for eqs. (2.8) and (2.27), we obtain

$$NE[\frac{\Delta f(\omega)}{f(\omega)} \cdot \frac{\Delta f(\eta)}{f(\eta)}] = K_4\ \sigma^{-4} + (f(\omega)\ f(\eta))^{-1} \sum_{k=-L-n}^{L+n}\ \sum_{k'=-L-n}^{L+n} g_k(\omega)$$
$$g_{k'}(\eta) \sum_{i=-\infty}^{\infty} R_i\ [R_{k+k'-i} + R_{k-k'-i}].$$

This is a final result to evaluate the CVE of SPE.

3 - NUMERICAL EXAMPLE

3 - 1. Formulas of CVE

We deduce the CVE of SPE at chapter 2. This CVE is compared with other two CVE in SPE. One of them is based on a method to solve first n simultaneous equations for having AR. The other one consists of the use of high order approximations of AR processes as acceptable values for ARMA processes.

We will evaluate the CVE of SPE found in chapter 2 and then compare this CVE with other two CVE. First, we show the method to solove simultaneous equations by the Yule-Walker equation. The CVE of SPE is given by

$$NE\left[\frac{\Delta f(\omega)}{f(\omega)} \cdot \frac{\Delta f(\eta)}{f(\eta)}\right] \simeq K_4 \ \sigma^{-4} + f(\omega)f(\eta)^{-1} \sum_{k=-2n+1}^{2n-1} \sum_{k'=-2n+1}^{2n-1} \sum_{i=-\infty}^{\infty}$$

(3.1)

$$g_k(\omega)g_{k'}(\eta)R_i(R_{k+k'-i} + R_{k-k'-i}) \qquad [8]$$

Second, when we estimate SPE by high order AR approximations for ARMA processes, we want to show :

$$NE\left[\frac{\Delta f(\omega)}{f(\omega)} \cdot \frac{\Delta f(\eta)}{f(\eta)}\right] \simeq K_4 \ \sigma^{-4} + 2 + \sigma^2 \ \underset{\sim}{L}'(\omega) \ R^{-1}(\eta)\underset{\sim}{L}(\eta)$$

(3.2) where $\underset{\sim}{L}'(\omega) = (L_1(\omega), L_2(\omega), \ldots, L_n(\omega))$

and $L_k(\omega) = 2R_e \left[\dfrac{e^{-jk(\omega)}}{1+c_1e^{-j\omega} + \ldots + c_me^{-jm\omega}}\right]$ [9].

It is difficult to find general properties of our equation in CVE. In other methods we also find very complex equations as above. Since we can not find statistical properties from CVE equations, a numerical example is carried out.

3 - 2. The evaluation of CVE

Let's evaluate CVE at $\omega = \eta$ angular frequencies on the example of an ARMA (2,1) model. An ARMA model is

$$y_t - 0.25\, y_{t-1} + 0.1\, y_{t-2} = u_t + 0.1\, u_{t-1}$$

where u_t is white noise.

In this case, CVE are computed by different sample time series of 300 sets the length of each one being 1500, and the average of the CVE of 300 sets sample time series is obtained. The results of computations are shown in Figure 1.

If the order of ARMA processes is sufficiently high, then it is evident that

$$\text{(3.3)} \qquad NE\left[\frac{\Delta f(s)}{f(s)} \cdot \frac{\Delta f(t)}{f(t)}\right] \simeq \begin{cases} 0 & s \neq t \\ 2m(1+\delta_{0,s} + \delta_{\pi,s}) & s = t \end{cases} \qquad [10] .$$

CVE is rather large at $s = 0$ and $s = \pi$ in all methods. The result of eq. (2.29) is similar to the result of eq. (3.1). It is evident that both results of eqs. (2.29) and (3.1) are smaller than the result of eq. (3.2). When we compare $L = 40$ with $L = 10$ and $L = 20$ in eq. (2.29), we obtain almost the same results. From this fact, we may choose rather freely as to the value of L. In this example, CVE is stable in all frequencies.

3 - 3. The conclusion of numerical examples

1° It is difficult to find general properties in eqs. (2.29), (3.1) and (3.2).

2° Results of eqs. (2.29) and (3.1) are almost the same, and both are smaller than results of eq. (3.2).

3° We can choose freely the value of L in eq. (2.29) (L = 10, 20, 40).

4° CVE in SPE is rather large near $\omega = 0$ and $\omega = \pi$.

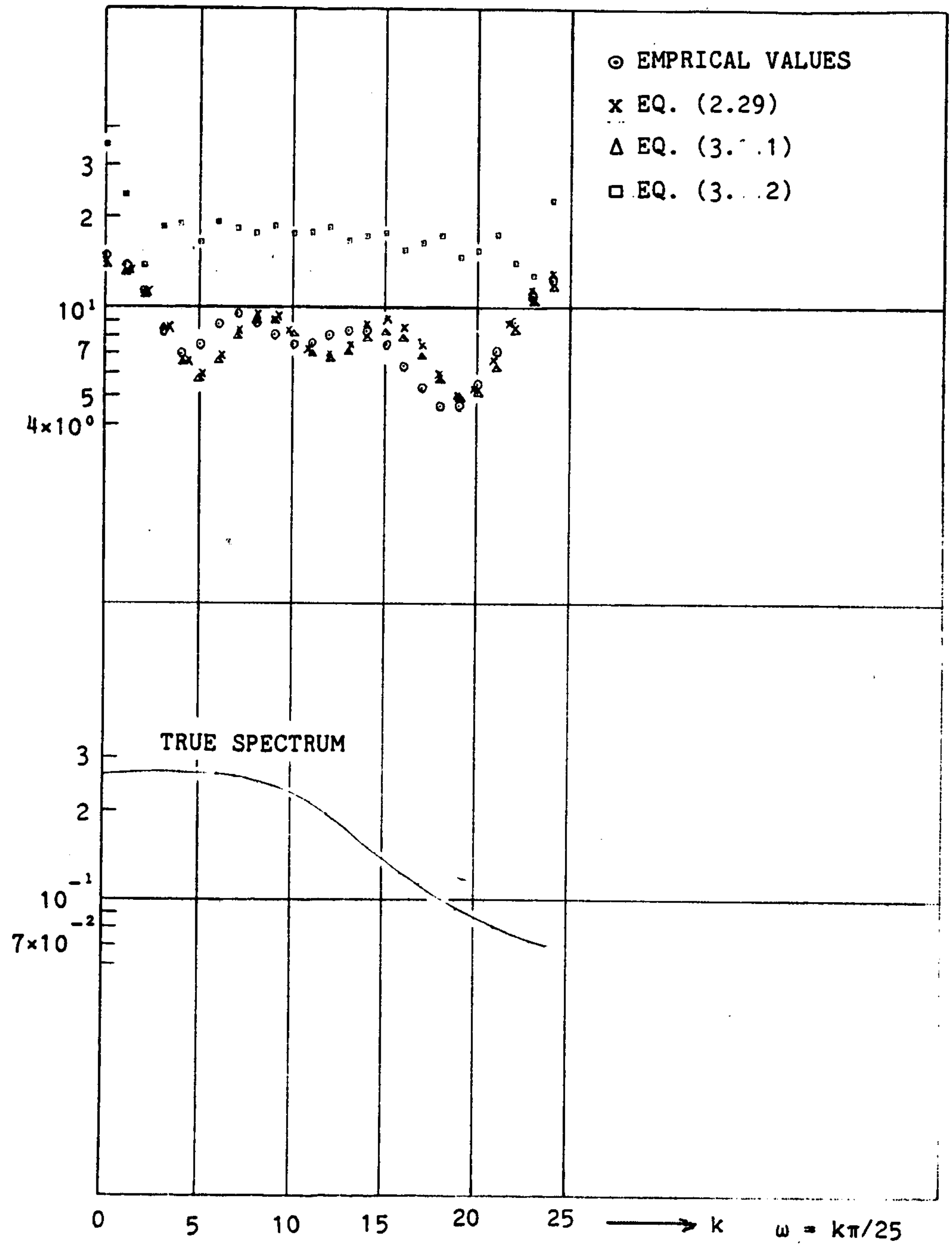

Figure 1 : VARIOUS CVE OF SPE

4.- CONCLUSION

This paper presents two results. First, the mathematical equation of the CVE in SPE for ARMA processes has been obtained. Second, the equation has been compared with other two CVEs in the SPE on our example. By means of the periodogram technique, one avoids using $\Delta\alpha$ for the calculation of the CVE in our theory.

This is a remarkable fact in the CVE of SPE for ARMA processes. From the result of our example, the result of the evaluation in eq. (2.29) is similar to the result of eq. (3.11). In the near future, we will study errors of SPE in multi-ARMA processes.

APPENDIX 1

OBTAINING A STATE SPACE MODEL AND $Q(\omega)$

We consider an ARMA (m, m-1) process for the time series y_s. An ARMA process is given by

$$y_s + \alpha_1 y_{s-1} + \dots + \alpha_m y_{s-m} = u_s + \beta_1 u_{s-1} + \dots + \beta_{m-1} u_{s-m+1}$$

(A1-1) and then it follows the state space model :

$$x_{s+1} = Ax_s + Bu_{s+1}$$

$$y_s = C' x_s$$

where x_s is an $n \times 1$ vector and the input vector u_s is an $r \times 1$ stationary white noise. In eq. (A1-1), the noises $u_s^1, u_s^2, \dots, u_s^r$ are mutually independent. The characteristic equation is written :

(A1-2) $$\det[\lambda I - A] = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n = 0$$

where I is the unit matrix and the roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$. We assume that the roots of eq. (A1-1) lie within the unit circle. When absolute values of all these roots are smaller than 1, the eq. (A1-1) of a system is asymptotically stable. Eliminating x_s in eq. (A1-1), the following equation holds :

$$\text{(A1-3)} \qquad y_{s+n} + \alpha_1 y_{s+n-1} + \ldots + \alpha_n y_s = \gamma'_0 u_{s+n} + \gamma'_1 u_{s+n-1} + \ldots + \gamma'_{n-1} u_{s+1}$$

where the γ' is an $r \times 1$ vector, defined as follows :

$$\text{(A1-4)} \qquad \begin{aligned} \gamma'_0 &= C'B \\ \gamma'_1 &= C'(A + \alpha_1 I)B \\ &\vdots \\ \gamma'_{n-1} &= C'(A^{n-1} + \alpha_1 A^{n-2} + \ldots + \alpha_{n-1} I)B. \end{aligned}$$

In eq. (A1-1), the relations between u_s and $x_{s-\ell}$ ($\ell = 0, 1, 2, \ldots$) are uncorrelated. From this fact, the autocovariance R_ℓ of y_s is given by

$$\text{(A1-5)} \qquad R_\ell \equiv E[y_s \; y_{s+\ell}] = C' A^\ell E[x_t \; x'_t]C$$

Then we obtain the following Yule-Walker equation

$$\text{(A1-6)} \qquad R_{n+\ell} + \alpha_1 R_{N+\ell-1} + \ldots + \alpha_n R_\ell = 0, \qquad \ell = 0, 1, 2, \ldots$$

When we multiply the Yule-Walker equations by $e^{-j\omega\ell}$ and sum them from $\ell = 0$ to ∞, we obtain

$$\text{(A1-7)} \qquad \sum_{k=0}^{n} \alpha_k \, e^{j\omega(n-k)} \{Q(\omega) - \sum_{\ell=0}^{n-k-1} R_\ell \, e^{-j\omega\ell}\} = 0.$$

From this equation, we will obtain the formula for $Q(\omega)$.

APPENDIX 2

ON THE SECOND TERM OF EQ. (2.24)

From eq. (2.14), it follows that

$$E[I_N(s)] = \frac{1}{2\pi} \sum_{k=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) R_k\, e^{-jks} ds. \tag{A2-1}$$

If the equation

$$\int_{-\pi}^{\pi} G(s,\omega) \frac{1}{2\pi} \sum_{k=-N+1}^{N-1} R_k\, e^{-jks} ds = f(\omega) \tag{A2-2}$$

is satisfied, then we can prove for N enough large, that the order of the second terms of eq. (2.24) becomes N^{-2} and thus these terms can be neglected.

From eq. (2.21), the following equation can be derived :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K(s,\omega) \sum_{k=-N+1}^{N-1} R_k\, e^{-jks}\, ds$$

$$= \sum_{i=1}^{n} \sum_{\ell=1}^{L+1} M_i(\omega)\, R_{\ell-2+i} \sum_{i'=0}^{n} \alpha_{i'} \sum_{k=-N+1}^{N-1} R_k \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(\ell+n-1-i'-k)s} ds \tag{A2-3}$$

$$= \sum_{i=1}^{n} \sum_{\ell=1}^{L+1} M_i(\omega)\, R_{\ell-2+i} \sum_{i'=0}^{n} \alpha_{i'}\, R_{\ell+n-1-i'}\,.$$

By the Yule-Walker equation (A1-6) $\sum_{i'=0}^{n} \alpha_{i'} R_{\ell+n-1-i'}$ is equal to zero, and hence the result finally becomes equal to zero.

We can obtain the equation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} T(s,\omega) \sum_{k=-N+1}^{N-1} R_k\, e^{-jks} ds = Q(\omega) \tag{A2-4}$$

by eqs. (2.10) and (2.19), and finally eq. (A2-2) is obtained.

When $N \to \infty$, eq. (A.2.2) can be written

$$\int_{-\pi}^{\pi} G(s,\omega)\, f(s)ds = f(\omega). \tag{A2-5}$$

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